Exercise 12

Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

$$y'' + 4y = e^{-x}$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \to \quad y_c' = re^{rx} \quad \to \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4 = 0$$

Solve for r.

$$r = \{-2i, 2i\}$$

Two solutions to the ODE are e^{-2ix} and e^{2ix}. By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{-2ix} + C_2 e^{2ix} \\ &= C_1(\cos 2x - i \sin 2x) + C_2(\cos 2x + i \sin 2x) \\ &= (C_1 + C_2)\cos 2x + (-iC_1 + iC_2)\sin 2x \\ &= C_3 \cos 2x + C_4 \sin 2x. \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + 4y_p = e^{-x} \tag{2}$$

Since the inhomogeneous term is an exponential function, the particular solution is $y_p = Ae^{-x}$.

$$y_p = Ae^{-x} \quad \to \quad y_p' = -Ae^{-x} \quad \to \quad y_p'' = Ae^{-x}$$

Substitute these formulas into equation (2).

$$(Ae^{-x}) + 4(Ae^{-x}) = e^{-x}$$

$$(A + 4A)e^{-x} = e^{-x}$$

Match the coefficients on both sides to get an equation for A.

$$A + 4A = 1$$

Solving it yields

$$A = \frac{1}{5},$$

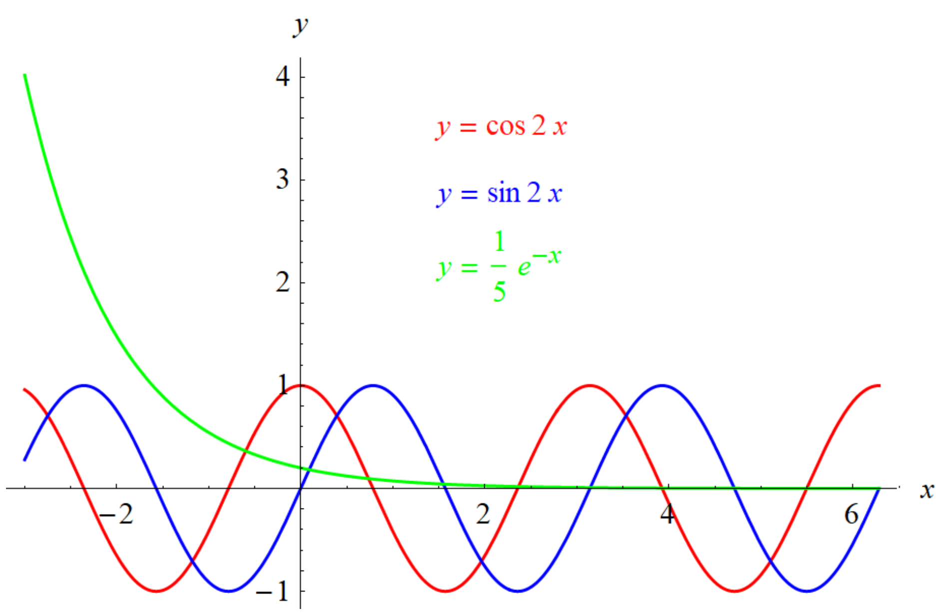
which means the particular solution is

$$y_p = \frac{1}{5}e^{-x}.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_3 \cos 2x + C_4 \sin 2x + \frac{1}{5}e^{-x}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants. Below is a graph of the two sinusoidal functions and the particular solution.



As $x \to \infty$, all solutions tend to a linear combination of $\cos 2x$ and $\sin 2x$. And as $x \to -\infty$, $y(x)$ blows up to infinity.